2.5 Limits Involving Infinity

In this section we investigate the global behavior of functions and, in particular, whether their graphs approach asymptotes, vertical or horizontal.

Infinite Limits

In Example 8 in Section 2.2 we concluded that

$$\lim_{x \to 0} \frac{1}{x^2} \quad \text{does not exist}$$

by observing from the table of values and the graph of $y = 1/x^2$ in Figure 1, that the values of $1/x^2$ can be made arbitrarily large by taking x close enough to 0. Thus, the values of $f(x)$ do not approach a number, so $\lim_{x \to 0} (1/x^2)$ does not exist.

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000

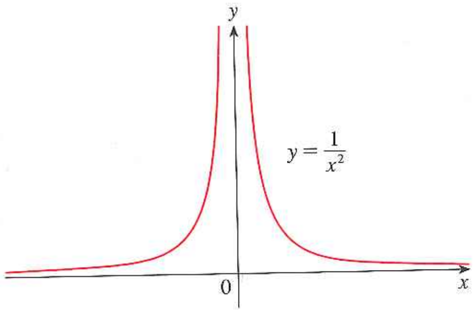


FIGURE 1

To indicate this kind of behavior we use the notation

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

This does not mean that we are regarding ∞ as a number. Nor does it mean that the limit exists. It simply expresses the particular way in which the limit does not exist: $1/x^2$ can be made as large as we like by taking x close enough to 0.

In general, we write symbolically

$$\lim_{x \to a} f(x) = \infty$$

to indicate that the values of $f(x)$ become larger and larger (or "increase without bound") as x approaches a.

1 Definition The notation

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a (on either side of a) but not equal to a.

■■ A more precise version of Definition 1 is given in Appendix D, Exercise 16.

Another notation for $\lim_{x \to a} f(x) = \infty$ is

$$f(x) \to \infty \qquad \text{as} \qquad x \to a$$

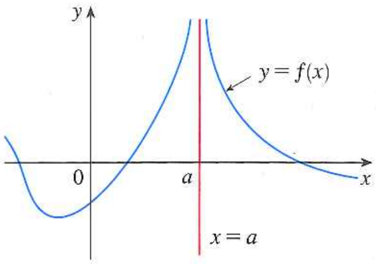


FIGURE 2

$\lim_{x \to a} f(x) = \infty$

■■ When we say that a number is "large negative," we mean that it is negative but its magnitude (absolute value) is large.

Again, the symbol ∞ is not a number, but the expression $\lim_{x \to a} f(x) = \infty$ is often read as

"the limit of $f(x)$, as x approaches a, is infinity"

or "$f(x)$ becomes infinite as x approaches a"

or "$f(x)$ increases without bound as x approaches a"

This definition is illustrated graphically in Figure 2.

Similarly, as shown in Figure 3,

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ are as large negative as we like for all values of x that are sufficiently close to a, but not equal to a.

The symbol $\lim_{x \to a} f(x) = -\infty$ can be read as "the limit of $f(x)$, as x approaches a, is negative infinity" or "$f(x)$ decreases without bound as x approaches a." As an example we have

$$\lim_{x \to 0} \left(-\frac{1}{x^2} \right) = -\infty$$

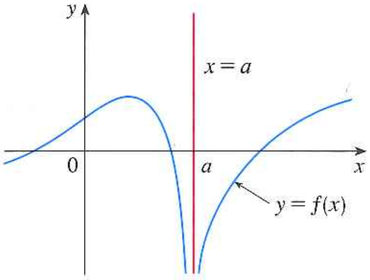


FIGURE 3

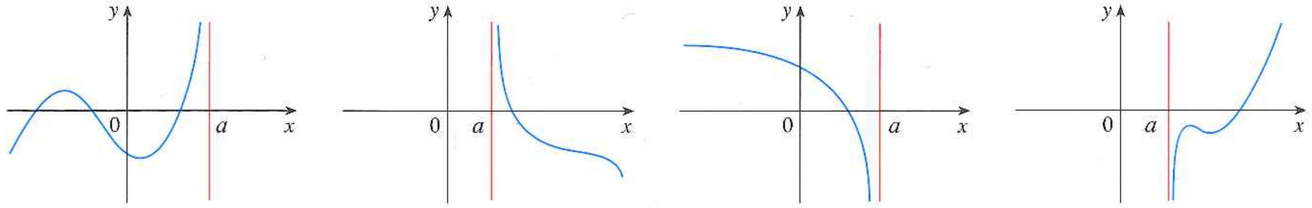
$\lim_{x \to a} f(x) = -\infty$

Similar definitions can be given for the one-sided infinite limits

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

remembering that "$x \to a^-$" means that we consider only values of x that are less than a, and similarly "$x \to a^+$" means that we consider only $x > a$. Illustrations of these four cases are given in Figure 4.



(a) $\lim_{x \to a^-} f(x) = \infty$ (b) $\lim_{x \to a^+} f(x) = \infty$ (c) $\lim_{x \to a^-} f(x) = -\infty$ (d) $\lim_{x \to a^+} f(x) = -\infty$

FIGURE 4

2 Definition The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

For instance, the y-axis is a vertical asymptote of the curve $y = 1/x^2$ because $\lim_{x \to 0} (1/x^2) = \infty$. In Figure 4 the line $x = a$ is a vertical asymptote in each of the four cases shown.

EXAMPLE 1 Find $\lim_{x\to 3^+} \dfrac{2x}{x-3}$ and $\lim_{x\to 3^-} \dfrac{2x}{x-3}$.

SOLUTION If x is close to 3 but larger than 3, then the denominator $x - 3$ is a small positive number and $2x$ is close to 6. So the quotient $2x/(x - 3)$ is a large *positive* number. Thus, intuitively we see that

$$\lim_{x\to 3^+} \frac{2x}{x-3} = \infty$$

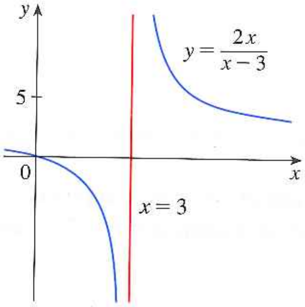


FIGURE 5

Likewise, if x is close to 3 but smaller than 3, then $x - 3$ is a small negative numb but $2x$ is still a positive number (close to 6). So $2x/(x - 3)$ is a numerically large *negative* number. Thus

$$\lim_{x\to 3^-} \frac{2x}{x-3} = -\infty$$

The graph of the curve $y = 2x/(x - 3)$ is given in Figure 5. The line $x = 3$ is a vertical asymptote.

Two familiar functions whose graphs have vertical asymptotes are $y = \tan x$ an $y = \ln x$. From Figure 6 we see that

$$\boxed{3} \qquad \boxed{\lim_{x\to 0^+} \ln x = -\infty}$$

and so the line $x = 0$ (the y-axis) is a vertical asymptote. In fact, the same is true fo $y = \log_a x$ provided that $a > 1$. (See Figures 11 and 12 in Section 1.6.)

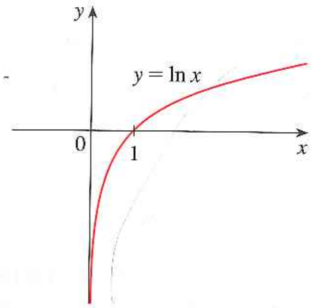


FIGURE 6

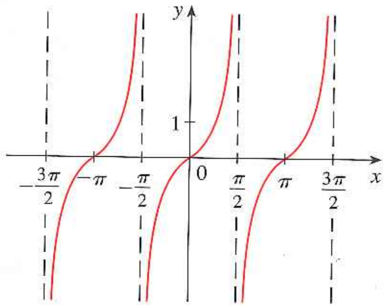


FIGURE 7
$y = \tan x$

Figure 7 shows that

$$\lim_{x\to(\pi/2)^-} \tan x = \infty$$

and so the line $x = \pi/2$ is a vertical asymptote. In fact, the lines $x = (2n + 1)\pi/2$, n an integer, are all vertical asymptotes of $y = \tan x$.

EXAMPLE 2 Find $\lim_{x\to 0} \ln(\tan^2 x)$.

SOLUTION We introduce a new variable, $t = \tan^2 x$. Then $t \geqslant 0$ and $t = \tan^2 x \to \tan^2 0 = 0$ as $x \to 0$ because tan is a continuous function. So, by (3), we have

$$\lim_{x\to 0} \ln(\tan^2 x) = \lim_{t\to 0^+} \ln t = -\infty$$

■■ The problem-solving strategy for Example 2 is *Introduce Something Extra* (see page 86). Here, the something extra, the auxiliary aid, is the new variable t.

Limits at Infinity

Limits at Infinity

In computing infinite limits, we let x approach a number and the result was that the values of y became arbitrarily large (positive or negative). Here we let x become arbitrarily large (positive or negative) and see what happens to y.

Let's begin by investigating the behavior of the function f defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

as x becomes large. The table at the left gives values of this function correct to six decimal places, and the graph of f has been drawn by a computer in Figure 8.

x	$f(x)$
0	−1
±1	0
±2	0.600000
±3	0.800000
±4	0.882353
±5	0.923077
±10	0.980198
±50	0.999200
±100	0.999800
±1000	0.999998

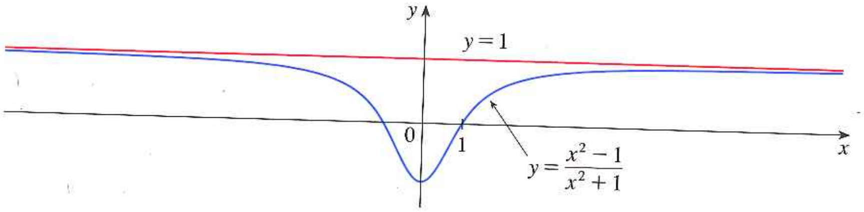


FIGURE 8

As x grows larger and larger you can see that the values of $f(x)$ get closer and closer to 1. In fact, it seems that we can make the values of $f(x)$ as close as we like to 1 by taking x sufficiently large. This situation is expressed symbolically by writing

$$\lim_{x\to\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

In general, we use the notation

$$\lim_{x\to\infty} f(x) = L$$

to indicate that the values of $f(x)$ approach L as x becomes larger and larger.

4 Definition Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that the values of $f(x)$ can be made as close to L as we like by taking x sufficiently large.

■■ A more precise version of Definition 4 is given in Appendix D.

Another notation for $\lim_{x\to\infty} f(x) = L$ is

$$f(x) \to L \quad \text{as} \quad x \to \infty$$

The symbol ∞ does not represent a number. Nonetheless, the expression $\lim_{x\to\infty} f(x) = L$ is often read as

"the limit of $f(x)$, as x approaches infinity, is L"

or "the limit of $f(x)$, as x becomes infinite, is L"

or "the limit of $f(x)$, as x increases without bound, is L"

The meaning of such phrases is given by Definition 4.

Geometric illustrations of Definition 4 are shown in Figure 9. Notice that there are many ways for the graph of f to approach the line $y = L$ (which is called a *horizontal asymptote*) as we look to the far right of each graph.

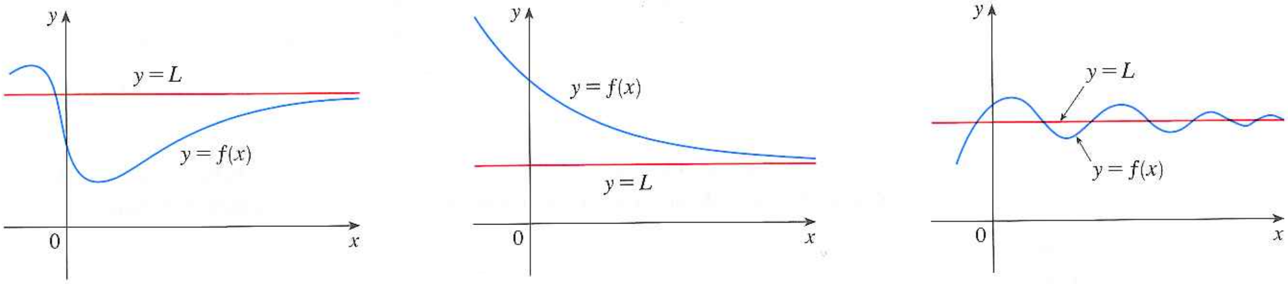


FIGURE 9

Examples illustrating $\lim_{x\to\infty} f(x) = L$

Referring back to Figure 8, we see that for numerically large negative values of x, the values of $f(x)$ are close to 1. By letting x decrease through negative values without bound, we can make $f(x)$ as close to 1 as we like. This is expressed by writing

$$\lim_{x\to-\infty} \frac{x^2-1}{x^2+1} = 1$$

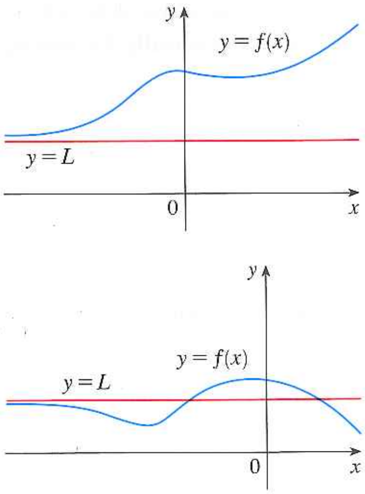


FIGURE 10

Examples illustrating $\lim_{x\to-\infty} f(x) = L$

In general, as shown in Figure 10, the notation

$$\lim_{x\to-\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

Again, the symbol $-\infty$ does not represent a number, but the expression $\lim_{x\to-\infty} f(x) = L$ is often read as

"the limit of $f(x)$, as x approaches negative infinity, is L"

5 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L$$

For instance, the curve illustrated in Figure 8 has the line $y = 1$ as a horizontal asymptote because

$$\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = 1$$

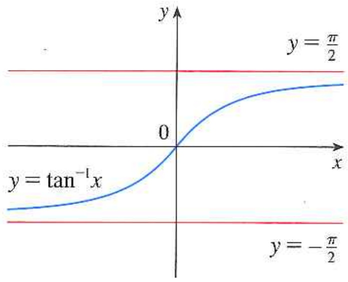


FIGURE 11

An example of a curve with two horizontal asymptotes is $y = \tan^{-1}x$. (See Figure 11.) In fact,

6

$$\boxed{\lim_{x \to -\infty} \tan^{-1}x = -\frac{\pi}{2} \qquad \lim_{x \to \infty} \tan^{-1}x = \frac{\pi}{2}}$$

so both of the lines $y = -\pi/2$ and $y = \pi/2$ are horizontal asymptotes. (This follows from the fact that the lines $x = \pm\pi/2$ are vertical asymptotes of the graph of tan.)

EXAMPLE 3 Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown in Figure 12.

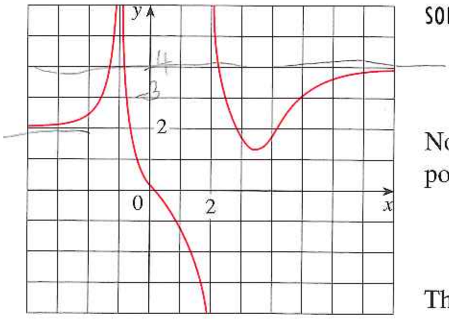


FIGURE 12

SOLUTION We see that the values of $f(x)$ become large as $x \to -1$ from both sides, so

$$\lim_{x \to -1} f(x) = \infty$$

Notice that $f(x)$ becomes large negative as x approaches 2 from the left, but large positive as x approaches 2 from the right. So

$$\lim_{x \to 2^-} f(x) = -\infty \qquad \text{and} \qquad \lim_{x \to 2^+} f(x) = \infty$$

Thus, both of the lines $x = -1$ and $x = 2$ are vertical asymptotes.

As x becomes large, it appears that $f(x)$ approaches 4. But as x decreases through negative values, $f(x)$ approaches 2. So

$$\lim_{x \to \infty} f(x) = 4 \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = 2$$

This means that both $y = 4$ and $y = 2$ are horizontal asymptotes. ■■

EXAMPLE 4 Find $\displaystyle\lim_{x \to \infty} \frac{1}{x}$ and $\displaystyle\lim_{x \to -\infty} \frac{1}{x}$.

SOLUTION Observe that when x is large, $1/x$ is small. For instance,

$$\frac{1}{100} = 0.01 \qquad \frac{1}{10{,}000} = 0.0001 \qquad \frac{1}{1{,}000{,}000} = 0.000001$$

In fact, by taking x large enough, we can make $1/x$ as close to 0 as we please. Therefore, according to Definition 4, we have

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Similar reasoning shows that when x is large negative, $1/x$ is small negative, so we also have

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

It follows that the line $y = 0$ (the x-axis) is a horizontal asymptote of the curve $y = 1/x$. (This is an equilateral hyperbola; see Figure 13.) ■■

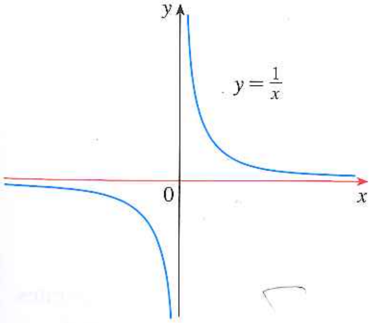


FIGURE 13

$\displaystyle\lim_{x \to \infty} \frac{1}{x} = 0, \quad \lim_{x \to -\infty} \frac{1}{x} = 0$

Most of the Limit Laws that were given in Section 2.3 also hold for limits at infinity. It can be proved that the *Limit Laws listed in Section 2.3 (with the exception of Laws 9 and 10) are also valid if "$x \to a$" is replaced by "$x \to \infty$" or "$x \to -\infty$."* In particular, if we combine Law 6 with the results of Example 4 we obtain the following important rule for calculating limits.

7 If n is a positive integer, then

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \qquad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

V **EXAMPLE 5** Evaluate

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

SOLUTION As x becomes large, both numerator and denominator become large, so it isn't obvious what happens to their ratio. We need to do some preliminary algebra.

To evaluate the limit at infinity of any rational function, we first divide both the numerator and denominator by the highest power of x that occurs in the denominator. (We may assume that $x \neq 0$, since we are interested only in large values of x.) In this case the highest power of x is x^2, and so, using the Limit Laws, we have

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{\dfrac{3x^2 - x - 2}{x^2}}{\dfrac{5x^2 + 4x + 1}{x^2}} = \lim_{x\to\infty} \frac{3 - \dfrac{1}{x} - \dfrac{2}{x^2}}{5 + \dfrac{4}{x} + \dfrac{1}{x^2}}$$

$$= \frac{\displaystyle\lim_{x\to\infty}\left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{\displaystyle\lim_{x\to\infty}\left(5 + \frac{4}{x} + \frac{1}{x^2}\right)}$$

$$= \frac{\displaystyle\lim_{x\to\infty} 3 - \lim_{x\to\infty}\frac{1}{x} - 2\lim_{x\to\infty}\frac{1}{x^2}}{\displaystyle\lim_{x\to\infty} 5 + 4\lim_{x\to\infty}\frac{1}{x} + \lim_{x\to\infty}\frac{1}{x^2}}$$

$$= \frac{3 - 0 - 0}{5 + 0 + 0} \qquad \text{[by (7)]}$$

$$= \frac{3}{5}$$

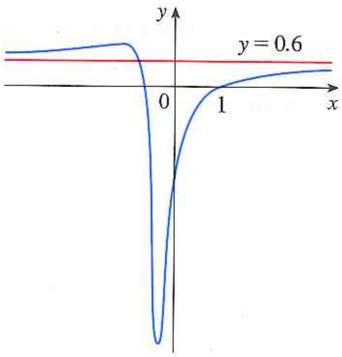


FIGURE 14
$y = \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$

A similar calculation shows that the limit as $x \to -\infty$ is also $\frac{3}{5}$. Figure 14 illustrates the results of these calculations by showing how the graph of the given rational function approaches the horizontal asymptote $y = \frac{3}{5}$. ■■

EXAMPLE 6 Compute $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$.

SOLUTION Because both $\sqrt{x^2+1}$ and x are large when x is large, it's difficult to see what happens to their difference, so we use algebra to rewrite the function. We first multiply numerator and denominator by the conjugate radical:

■■ We can think of the given function as having a denominator of 1.

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$

$$=\lim_{x\to\infty}\frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x}=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}$$

The Squeeze Theorem could be used to show that this limit is 0. But an easier method is to divide numerator and denominator by x. Doing this and remembering that $x=\sqrt{x^2}$ for $x>0$, we obtain

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}=\lim_{x\to\infty}\frac{\dfrac{1}{x}}{\dfrac{\sqrt{x^2+1}+x}{x}}$$

$$=\lim_{x\to\infty}\frac{\dfrac{1}{x}}{\sqrt{1+\dfrac{1}{x^2}}+1}=\frac{0}{\sqrt{1+0}+1}=0$$

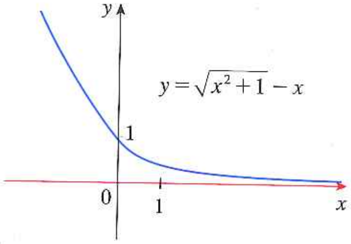


FIGURE 15

Figure 15 illustrates this result. ■■

The graph of the natural exponential function $y=e^x$ has the line $y=0$ (the x-axis) as a horizontal asymptote. (The same is true of any exponential function with base $a>1$.) In fact, from the graph in Figure 16 and the corresponding table of values, we see that

$$\boxed{8}\qquad\boxed{\lim_{x\to-\infty}e^x=0}$$

Notice that the values of e^x approach 0 very rapidly.

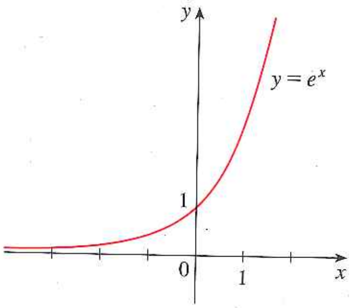


FIGURE 16

x	e^x
0	1.00000
−1	0.36788
−2	0.13534
−3	0.04979
−5	0.00674
−8	0.00034
−10	0.00005

V **EXAMPLE 7** Evaluate $\lim_{x \to 0^-} e^{1/x}$.

SOLUTION If we let $t = 1/x$, we know from Example 4 that $t \to -\infty$ as $x \to 0^-$. Therefore, by (8),

$$\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0$$

EXAMPLE 8 Evaluate $\lim_{x \to \infty} \sin x$.

SOLUTION As x increases, the values of $\sin x$ oscillate between 1 and -1 infinitely often. Thus, $\lim_{x \to \infty} \sin x$ does not exist.

Infinite Limits at Infinity

The notation

$$\lim_{x \to \infty} f(x) = \infty$$

is used to indicate that the values of $f(x)$ become large as x becomes large. Similar meanings are attached to the following symbols:

$$\lim_{x \to -\infty} f(x) = \infty \qquad \lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

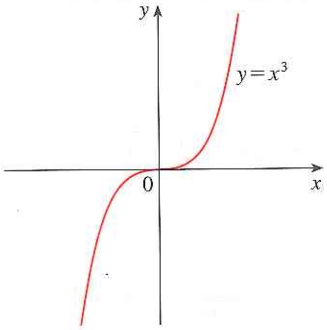


FIGURE 17

From Figures 16 and 17 we see that

$$\lim_{x \to \infty} e^x = \infty \qquad \lim_{x \to \infty} x^3 = \infty \qquad \lim_{x \to -\infty} x^3 = -\infty$$

but, as Figure 18 demonstrates, $y = e^x$ becomes large as $x \to \infty$ at a much faster rate than $y = x^3$.

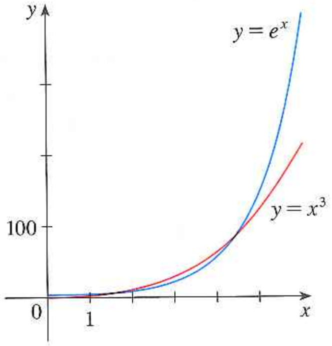


FIGURE 18

EXAMPLE 9 Find $\lim_{x \to \infty} (x^2 - x)$.

SOLUTION It would be **wrong** to write

$$\lim_{x \to \infty} (x^2 - x) = \lim_{x \to \infty} x^2 - \lim_{x \to \infty} x = \infty - \infty$$

The Limit Laws can't be applied to infinite limits because ∞ is not a number ($\infty - \infty$ can't be defined). However, we can write

$$\lim_{x \to \infty} (x^2 - x) = \lim_{x \to \infty} x(x - 1) = \infty$$

because both x and $x - 1$ become arbitrarily large.

$-\infty$, 1/6, 2, 0, ∞
1/2, DNE, $-\infty$

EXAMPLE 10 Find $\lim_{x\to\infty} \dfrac{x^2+x}{3-x}$.

SOLUTION We divide numerator and denominator by x (the highest power of x that occurs in the denominator):

$$\lim_{x\to\infty} \frac{x^2+x}{3-x} = \lim_{x\to\infty} \frac{x+1}{\dfrac{3}{x}-1} = -\infty$$

because $x + 1 \to \infty$ and $3/x - 1 \to -1$ as $x \to \infty$. ■ ■

2.5 Exercises

1. Explain in your own words the meaning of each of the following.

(a) $\lim_{x\to 2} f(x) = \infty$ (b) $\lim_{x\to 1^+} f(x) = -\infty$

(c) $\lim_{x\to\infty} f(x) = 5$ (d) $\lim_{x\to -\infty} f(x) = 3$

2. (a) Can the graph of $y = f(x)$ intersect a vertical asymptote? Can it intersect a horizontal asymptote? Illustrate by sketching graphs.

(b) How many horizontal asymptotes can the graph of $y = f(x)$ have? Sketch graphs to illustrate the possibilities.

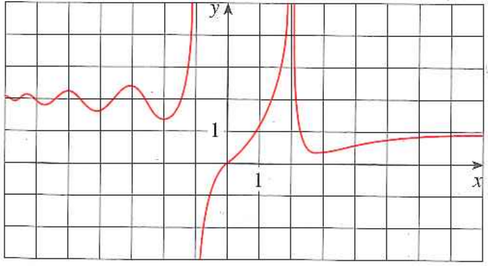
3. For the function f whose graph is given, state the following.

(a) $\lim_{x\to 2} f(x)$ (b) $\lim_{x\to -1^-} f(x)$

(c) $\lim_{x\to -1^+} f(x)$ (d) $\lim_{x\to\infty} f(x)$

(e) $\lim_{x\to -\infty} f(x)$ (f) The equations of the asymptotes

[illegible]

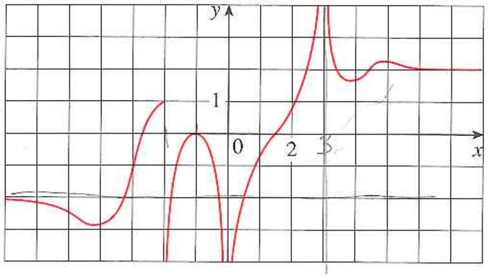


4. For the function g whose graph is given, state the following.

(a) $\lim_{x\to\infty} g(x)$ =3 (b) $\lim_{x\to -\infty} g(x)$ = 6

(c) $\lim_{x\to 3} g(x)$ ∞ (d) $\lim_{x\to 0} g(x)$ - $-\infty$

(e) $\lim_{x\to -2^+} g(x)$ (f) The equations of the asymptotes



5–10 ■ Sketch the graph of an example of a function f that satisfies all of the given conditions.

5. $f(0) = 0,\quad f(1) = 1,\quad \lim_{x\to\infty} f(x) = 0,\quad f$ is odd

6. $\lim_{x\to 0^+} f(x) = \infty,\quad \lim_{x\to 0^-} f(x) = -\infty,\quad \lim_{x\to\infty} f(x) = 1,$
$\lim_{x\to -\infty} f(x) = 1$

7. $\lim_{x\to 2} f(x) = -\infty,\quad \lim_{x\to\infty} f(x) = \infty,\quad \lim_{x\to -\infty} f(x) = 0,$
$\lim_{x\to 0^+} f(x) = \infty,\quad \lim_{x\to 0^-} f(x) = -\infty$

8. $\lim_{x\to -2} f(x) = \infty,\quad \lim_{x\to -\infty} f(x) = 3,\quad \lim_{x\to\infty} f(x) = -3$

9. $f(0) = 3,\quad \lim_{x\to 0^-} f(x) = 4,\quad \lim_{x\to 0^+} f(x) = 2,$
$\lim_{x\to -\infty} f(x) = -\infty,\quad \lim_{x\to 4^-} f(x) = -\infty,\quad \lim_{x\to 4^+} f(x) = \infty,$
$\lim_{x\to\infty} f(x) = 3$

10. $\lim_{x\to 3} f(x) = -\infty,\quad \lim_{x\to\infty} f(x) = 2,\quad f(0) = 0,\quad f$ is even