

2.3 Calculating Limits Using the Limit Laws

In Section 2.2 we used calculators and graphs to guess the values of limits, but we saw that such methods don't always lead to the correct answer. In this section we use the following properties of limits, called the *Limit Laws*, to calculate limits.

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \rightarrow a} f(x) \quad \text{and} \quad \lim_{x \rightarrow a} g(x)$$

exist. Then

$$1. \lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

$$2. \lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

$$3. \lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$$

$$4. \lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

$$5. \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{if } \lim_{x \rightarrow a} g(x) \neq 0$$

These five laws can be stated verbally as follows:

1. The limit of a sum is the sum of the limits.
2. The limit of a difference is the difference of the limits.
3. The limit of a constant times a function is the constant times the limit of the function.
4. The limit of a product is the product of the limits.
5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

It is easy to believe that these properties are true. For instance, if $f(x)$ is close to L and $g(x)$ is close to M , it is reasonable to conclude that $f(x) + g(x)$ is close to $L + M$. This gives us an intuitive basis for believing that Law 1 is true. All of these laws can be proved using the precise definition of a limit. In Appendix E we give the proof of Law 1.

EXAMPLE 1 Use the Limit Laws and the graphs of f and g in Figure 1 to evaluate the following limits, if they exist.

$$(a) \lim_{x \rightarrow -2} [f(x) + 5g(x)] \quad (b) \lim_{x \rightarrow 1} [f(x)g(x)] \quad (c) \lim_{x \rightarrow 2} \frac{f(x)}{g(x)}$$

SOLUTION

(a) From the graphs of f and g we see that

$$\lim_{x \rightarrow -2} f(x) = 1 \quad \text{and} \quad \lim_{x \rightarrow -2} g(x) = -1$$

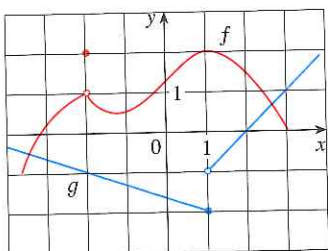


FIGURE 1

- Sum Law
- Difference Law
- Constant Multiple Law
- Product Law
- Quotient Law

Therefore, we have

$$\begin{aligned}\lim_{x \rightarrow -2} [f(x) + 5g(x)] &= \lim_{x \rightarrow -2} f(x) + \lim_{x \rightarrow -2} [5g(x)] && \text{(by Law 1)} \\ &= \lim_{x \rightarrow -2} f(x) + 5 \lim_{x \rightarrow -2} g(x) && \text{(by Law 3)} \\ &= 1 + 5(-1) = -4\end{aligned}$$

(b) We see that $\lim_{x \rightarrow 1} f(x) = 2$. But $\lim_{x \rightarrow 1} g(x)$ does not exist because the left and right limits are different:

$$\lim_{x \rightarrow 1^-} g(x) = -2 \quad \lim_{x \rightarrow 1^+} g(x) = -1$$

So we can't use Law 4 for the desired limit. But we *can* use Law 4 for the one-sided limits:

$$\lim_{x \rightarrow 1^-} [f(x)g(x)] = 2 \cdot (-2) = -4 \quad \lim_{x \rightarrow 1^+} [f(x)g(x)] = 2 \cdot (-1) = -2$$

The left and right limits aren't equal, so $\lim_{x \rightarrow 1} f(x)$ does not exist.

(c) The graphs show that

$$\lim_{x \rightarrow 2} f(x) \approx 1.4 \quad \text{and} \quad \lim_{x \rightarrow 2} g(x) = 0$$

Because the limit of the denominator is 0, we can't use Law 5. The given limit does not exist because the denominator approaches 0 while the numerator approaches a nonzero number. ■ ■

If we use the Product Law repeatedly with $g(x) = f(x)$, we obtain the following law.

Power Law

$$6. \lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n \quad \text{where } n \text{ is a positive integer}$$

In applying these six limit laws, we need to use two special limits:

$$7. \lim_{x \rightarrow a} c = c$$

$$8. \lim_{x \rightarrow a} x = a$$

These limits are obvious from an intuitive point of view (state them in words or draw graphs of $y = c$ and $y = x$).

If we now put $f(x) = x$ in Law 6 and use Law 8, we get another useful special limit.

$$9. \lim_{x \rightarrow a} x^n = a^n \quad \text{where } n \text{ is a positive integer}$$

A similar limit holds for roots as follows.

$$10. \lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a} \quad \text{where } n \text{ is a positive integer}$$

(If n is even, we assume that $a > 0$.)

Root Law

More generally, we have the following law.

$$\text{II. } \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} \quad \text{where } n \text{ is a positive integer}$$

[If n is even, we assume that $\lim_{x \rightarrow a} f(x) > 0$.]

■ ■ NEWTON AND LIMITS

Isaac Newton was born on Christmas Day in 1642, the year of Galileo's death. When he entered Cambridge University in 1661 Newton didn't know much mathematics, but he learned quickly by reading Euclid and Descartes and by attending the lectures of Isaac Barrow. Cambridge was closed because of the plague in 1665 and 1666, and Newton returned home to reflect on what he had learned. Those two years were amazingly productive for at that time he made four of his major discoveries: (1) his representation of functions as sums of infinite series, including the binomial theorem; (2) his work on differential and integral calculus; (3) his laws of motion and law of universal gravitation; and (4) his prism experiments on the nature of light and color. Because of a fear of controversy and criticism, he was reluctant to publish his discoveries and it wasn't until 1687, at the urging of the astronomer Halley, that Newton published *Principia Mathematica*. In this work, the greatest scientific treatise ever written, Newton set forth his version of calculus and used it to investigate mechanics, fluid dynamics, and wave motion, and to explain the motion of planets and comets.

The beginnings of calculus are found in the calculations of areas and volumes by ancient Greek scholars such as Eudoxus and Archimedes. Although aspects of the idea of a limit are implicit in their "method of exhaustion," Eudoxus and Archimedes never explicitly formulated the concept of a limit. Likewise, mathematicians such as Cavalieri, Fermat, and Barrow, the immediate precursors of Newton in the development of calculus, did not actually use limits. It was Isaac Newton who was the first to talk explicitly about limits. He explained that the main idea behind limits is that quantities "approach nearer than by any given difference." Newton stated that the limit was the basic concept in calculus, but it was left to later mathematicians like Cauchy to clarify his ideas about limits.

EXAMPLE 2 Evaluate the following limits and justify each step.

$$(a) \lim_{x \rightarrow 5} (2x^2 - 3x + 4) \qquad (b) \lim_{x \rightarrow -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

SOLUTION

$$\begin{aligned} (a) \quad \lim_{x \rightarrow 5} (2x^2 - 3x + 4) &= \lim_{x \rightarrow 5} (2x^2) - \lim_{x \rightarrow 5} (3x) + \lim_{x \rightarrow 5} 4 && \text{(by Laws 2 and 1)} \\ &= 2 \lim_{x \rightarrow 5} x^2 - 3 \lim_{x \rightarrow 5} x + \lim_{x \rightarrow 5} 4 && \text{(by 3)} \\ &= 2(5^2) - 3(5) + 4 && \text{(by 9, 8, and 7)} \\ &= 39 \end{aligned}$$

(b) We start by using Law 5, but its use is fully justified only at the final stage when we see that the limits of the numerator and denominator exist and the limit of the denominator is not 0.

$$\begin{aligned} \lim_{x \rightarrow -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} &= \frac{\lim_{x \rightarrow -2} (x^3 + 2x^2 - 1)}{\lim_{x \rightarrow -2} (5 - 3x)} && \text{(by Law 5)} \\ &= \frac{\lim_{x \rightarrow -2} x^3 + 2 \lim_{x \rightarrow -2} x^2 - \lim_{x \rightarrow -2} 1}{\lim_{x \rightarrow -2} 5 - 3 \lim_{x \rightarrow -2} x} && \text{(by 1, 2, and 3)} \\ &= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} && \text{(by 9, 8, and 7)} \\ &= -\frac{1}{11} \end{aligned}$$

NOTE • If we let $f(x) = 2x^2 - 3x + 4$, then $f(5) = 39$. In other words, we would have gotten the correct answer in Example 2(a) by substituting 5 for x . Similarly, direct substitution provides the correct answer in part (b). The functions in Example 2 are a polynomial and a rational function, respectively, and similar use of the Limit Laws proves that direct substitution always works for such functions (see Exercises 41 and 42). We state this fact as follows.

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f , then

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Functions with the Direct Substitution Property are called *continuous at a* and will be studied in Section 2.4. However, not all limits can be evaluated by direct substitution, as the following examples show.

EXAMPLE 3 Find $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$.

SOLUTION Let $f(x) = (x^2 - 1)/(x - 1)$. We can't find the limit by substituting $x = 1$ because $f(1)$ isn't defined. Nor can we apply the Quotient Law, because the limit of the denominator is 0. Instead, we need to do some preliminary algebra. We factor the numerator as a difference of squares:

$$\frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1}$$

The numerator and denominator have a common factor of $x - 1$. When we take the limit as x approaches 1, we have $x \neq 1$ and so $x - 1 \neq 0$. Therefore, we can cancel the common factor and compute the limit as follows:

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{x - 1} \\ &= \lim_{x \rightarrow 1} (x + 1) \\ &= 1 + 1 = 2 \end{aligned}$$

The limit in this example arose in Section 2.1 when we were trying to find the tangent to the parabola $y = x^2$ at the point $(1, 1)$. ■ ■

NOTE ■ In Example 3 we were able to compute the limit by replacing the given function $f(x) = (x^2 - 1)/(x - 1)$ by a simpler function, $g(x) = x + 1$, with the same limit. This is valid because $f(x) = g(x)$ except when $x = 1$, and in computing a limit as x approaches 1 we don't consider what happens when x is actually equal to 1. In general, we have the following useful fact.

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$, provided the limits exist.

EXAMPLE 4 Find $\lim_{x \rightarrow 1} g(x)$ where

$$g(x) = \begin{cases} x + 1 & \text{if } x \neq 1 \\ \pi & \text{if } x = 1 \end{cases}$$

SOLUTION Here g is defined at $x = 1$ and $g(1) = \pi$, but the value of a limit as x approaches 1 does not depend on the value of the function at 1. Since $g(x) = x + 1$ for $x \neq 1$, we have

$$\lim_{x \rightarrow 1} g(x) = \lim_{x \rightarrow 1} (x + 1) = 2 \quad \blacksquare \blacksquare$$

Note that the values of the functions in Examples 3 and 4 are identical except when $x = 1$ (see Figure 2) and so they have the same limit as x approaches 1.

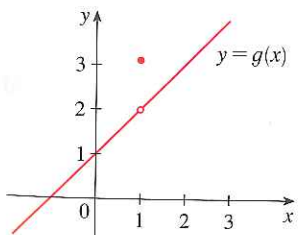
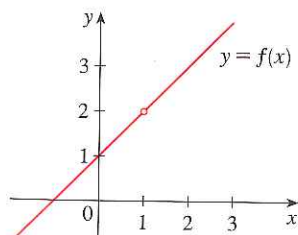


FIGURE 2

The graphs of the functions f (from Example 3) and g (from Example 4)

V **EXAMPLE 5** Evaluate $\lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h}$.

SOLUTION If we define

$$F(h) = \frac{(3+h)^2 - 9}{h}$$

then, as in Example 3, we can't compute $\lim_{h \rightarrow 0} F(h)$ by letting $h = 0$ since $F(0)$ is undefined. But if we simplify $F(h)$ algebraically, we find that

$$F(h) = \frac{(9 + 6h + h^2) - 9}{h} = \frac{6h + h^2}{h} = 6 + h$$

(Recall that we consider only $h \neq 0$ when letting h approach 0.) Thus

$$\lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h} = \lim_{h \rightarrow 0} (6 + h) = 6$$

EXAMPLE 6 Find $\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$.

SOLUTION We can't apply the Quotient Law immediately, since the limit of the denominator is 0. Here the preliminary algebra consists of rationalizing the numerator:

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} &= \lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} \cdot \frac{\sqrt{t^2 + 9} + 3}{\sqrt{t^2 + 9} + 3} \\ &= \lim_{t \rightarrow 0} \frac{(t^2 + 9) - 9}{t^2(\sqrt{t^2 + 9} + 3)} = \lim_{t \rightarrow 0} \frac{t^2}{t^2(\sqrt{t^2 + 9} + 3)} \\ &= \lim_{t \rightarrow 0} \frac{1}{\sqrt{t^2 + 9} + 3} = \frac{1}{\sqrt{\lim_{t \rightarrow 0} (t^2 + 9)} + 3} = \frac{1}{3 + 3} = \frac{1}{6} \end{aligned}$$

This calculation confirms the guess that we made in Example 2 in Section 2.2. ■ ■

Some limits are best calculated by first finding the left- and right-hand limits. The following theorem is a reminder of what we discovered in Section 2.2. It says that a two-sided limit exists if and only if both of the one-sided limits exist and are equal.

1 Theorem	$\lim_{x \rightarrow a} f(x) = L$ if and only if $\lim_{x \rightarrow a^-} f(x) = L = \lim_{x \rightarrow a^+} f(x)$
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When computing one-sided limits, we use the fact that the Limit Laws also hold for one-sided limits.

EXAMPLE 7 Show that $\lim_{x \rightarrow 0} |x| = 0$.

SOLUTION Recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

■ The result of Example 7 looks plausible from Figure 3.

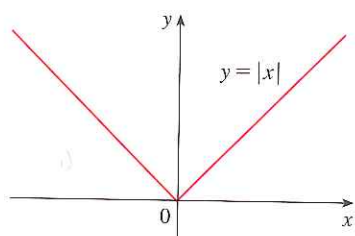


FIGURE 3

Since $|x| = x$ for $x > 0$, we have

$$\lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$ and so

$$\lim_{x \rightarrow 0^-} |x| = \lim_{x \rightarrow 0^-} (-x) = 0$$

Therefore, by Theorem 1,

$$\lim_{x \rightarrow 0} |x| = 0$$

▮ **EXAMPLE 8** Prove that $\lim_{x \rightarrow 0} \frac{|x|}{x}$ does not exist.

SOLUTION

$$\lim_{x \rightarrow 0^+} \frac{|x|}{x} = \lim_{x \rightarrow 0^+} \frac{x}{x} = \lim_{x \rightarrow 0^+} 1 = 1$$

$$\lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = \lim_{x \rightarrow 0^-} (-1) = -1$$

Since the right- and left-hand limits are different, it follows from Theorem 1 that $\lim_{x \rightarrow 0} |x|/x$ does not exist. The graph of the function $f(x) = |x|/x$ is shown in Figure 4 and supports the one-sided limits that we found.

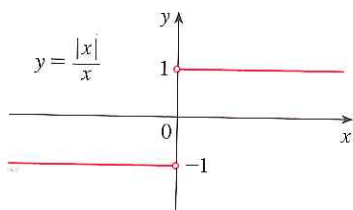


FIGURE 4

■ Other notations for $\llbracket x \rrbracket$ are $[x]$ and $\lfloor x \rfloor$. The greatest integer function is sometimes called the *floor function*.

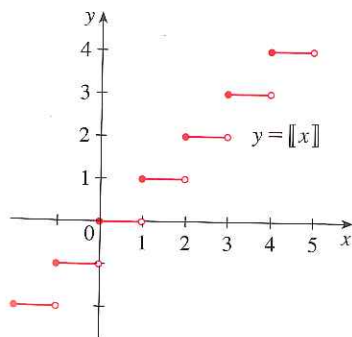


FIGURE 5
Greatest integer function

EXAMPLE 9 The **greatest integer function** is defined by $\llbracket x \rrbracket =$ the largest integer that is less than or equal to x . (For instance, $\llbracket 4 \rrbracket = 4$, $\llbracket 4.8 \rrbracket = 4$, $\llbracket \pi \rrbracket = 3$, $\llbracket \sqrt{2} \rrbracket = 1$, $\llbracket -\frac{1}{2} \rrbracket = -1$.) Show that $\lim_{x \rightarrow 3} \llbracket x \rrbracket$ does not exist.

SOLUTION The graph of the greatest integer function is shown in Figure 5. Since $\llbracket x \rrbracket = 3$ for $3 \leq x < 4$, we have

$$\lim_{x \rightarrow 3^+} \llbracket x \rrbracket = \lim_{x \rightarrow 3^+} 3 = 3$$

Since $\llbracket x \rrbracket = 2$ for $2 \leq x < 3$, we have

$$\lim_{x \rightarrow 3^-} \llbracket x \rrbracket = \lim_{x \rightarrow 3^-} 2 = 2$$

Because these one-sided limits are not equal, $\lim_{x \rightarrow 3} \llbracket x \rrbracket$ does not exist by Theorem 1.

The next two theorems give two additional properties of limits. Both can be proved using the precise definition of a limit in Appendix D.

2 Theorem If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a , then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$$

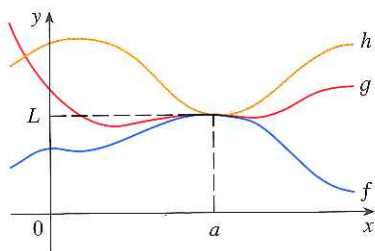


FIGURE 6

3 The Squeeze Theorem If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} g(x) = L$$

The Squeeze Theorem, which is sometimes called the Sandwich Theorem or the Pinching Theorem, is illustrated by Figure 6. It says that if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a , and if f and h have the same limit L at a , then g is forced to have the same limit L at a .

V EXAMPLE 10 Show that $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$.

SOLUTION First note that we **cannot** use

$$\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = \lim_{x \rightarrow 0} x^2 \cdot \lim_{x \rightarrow 0} \sin \frac{1}{x}$$

because $\lim_{x \rightarrow 0} \sin(1/x)$ does not exist (see Example 4 in Section 2.2). However, since

$$-1 \leq \sin \frac{1}{x} \leq 1$$

we have, as illustrated by Figure 7,

$$-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$$

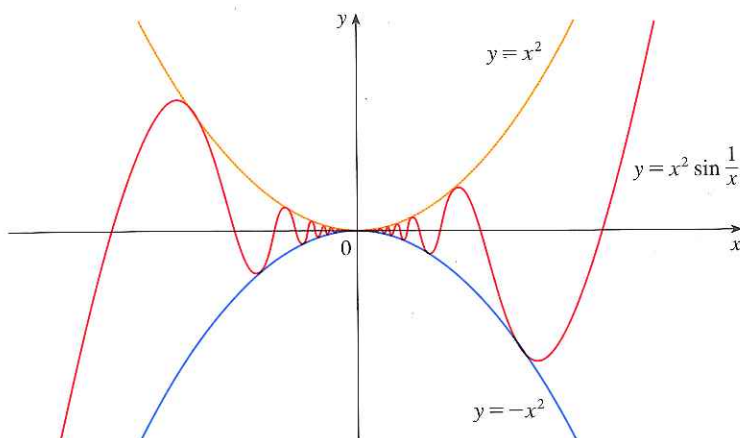


FIGURE 7

We know that

$$\lim_{x \rightarrow 0} x^2 = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (-x^2) = 0$$

Taking $f(x) = -x^2$, $g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain

$$\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$$

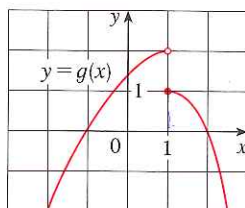
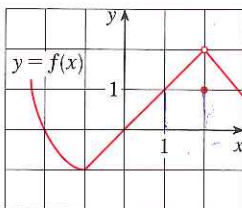
2.3 Exercises

1. Given that

$$\lim_{x \rightarrow a} f(x) = -3 \quad \lim_{x \rightarrow a} g(x) = 0 \quad \lim_{x \rightarrow a} h(x) = 8$$

find the limits that exist. If the limit does not exist, explain why.

- (a) $\lim_{x \rightarrow a} [f(x) + h(x)]$ (b) $\lim_{x \rightarrow a} [f(x)]^2$
 (c) $\lim_{x \rightarrow a} \sqrt[3]{h(x)}$ (d) $\lim_{x \rightarrow a} \frac{1}{f(x)}$
 (e) $\lim_{x \rightarrow a} \frac{f(x)}{h(x)}$ (f) $\lim_{x \rightarrow a} \frac{g(x)}{f(x)}$
 (g) $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ (h) $\lim_{x \rightarrow a} \frac{2f(x)}{h(x) - f(x)}$

2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.

- (a) $\lim_{x \rightarrow 2} [f(x) + g(x)]$ (b) $\lim_{x \rightarrow 1} [f(x) + g(x)]$
 (c) $\lim_{x \rightarrow 0} [f(x)g(x)]$ (d) $\lim_{x \rightarrow -1} \frac{f(x)}{g(x)}$
 (e) $\lim_{x \rightarrow 2} [x^3 f(x)]$ (f) $\lim_{x \rightarrow 1} \sqrt{3 + f(x)}$

3-7 ■ Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\lim_{x \rightarrow -2} (3x^4 + 2x^2 - x + 1)$ 4. $\lim_{t \rightarrow -1} (t^2 + 1)^3(t + 3)^5$
 5. $\lim_{x \rightarrow 8} (1 + \sqrt[3]{x})(2 - 6x^2 + x^3)$ 6. $\lim_{u \rightarrow -2} \sqrt{u^4 + 3u + 6}$
 7. $\lim_{x \rightarrow 1} \left(\frac{1 + 3x}{1 + 4x^2 + 3x^4} \right)^3$

8. (a) What is wrong with the following equation?

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

(b) In view of part (a), explain why the equation

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \rightarrow 2} (x + 3)$$

is correct.

9-22 ■ Evaluate the limit, if it exists.

9. $\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2}$ 10. $\lim_{x \rightarrow -4} \frac{x^2 + 5x + 4}{x^2 + 3x - 4}$
 11. $\lim_{x \rightarrow 2} \frac{x^2 - x + 6}{x - 2}$ 12. $\lim_{x \rightarrow 4} \frac{x^2 - 4x}{x^2 - 3x - 4}$
 13. $\lim_{t \rightarrow -3} \frac{t^2 - 9}{2t^2 + 7t + 3}$ 14. $\lim_{x \rightarrow -1} \frac{x^2 - 4x}{x^2 - 3x - 4}$
 15. $\lim_{h \rightarrow 0} \frac{(4 + h)^2 - 16}{h}$ 16. $\lim_{h \rightarrow 0} \frac{\sqrt{1 + h} - 1}{h}$
 17. $\lim_{x \rightarrow -2} \frac{x + 2}{x^3 + 8}$ 18. $\lim_{x \rightarrow -1} \frac{x^2 + 2x + 1}{x^4 - 1}$
 19. $\lim_{x \rightarrow 7} \frac{\sqrt{x + 2} - 3}{x - 7}$ 20. $\lim_{h \rightarrow 0} \frac{(3 + h)^{-1} - 3^{-1}}{h}$
 21. $\lim_{x \rightarrow -4} \frac{\frac{1}{4} + \frac{1}{x}}{4 + x}$ 22. $\lim_{t \rightarrow 0} \left(\frac{1}{t} - \frac{1}{t^2 + t} \right)$

23. (a) Estimate the value of

$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{1 + 3x} - 1}$$

- by graphing the function $f(x) = x/(\sqrt{1 + 3x} - 1)$.
 (b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.
 (c) Use the Limit Laws to prove that your guess is correct.

24. (a) Use a graph of

$$f(x) = \frac{\sqrt{3 + x} - \sqrt{3}}{x}$$

- to estimate the value of $\lim_{x \rightarrow 0} f(x)$ to two decimal places.
 (b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.
 (c) Use the Limit Laws to find the exact value of the limit.

25. Use the Squeeze Theorem to show that $\lim_{x \rightarrow 0} x^2 \cos 20\pi x = 0$. Illustrate by graphing the functions $f(x) = -x^2$, $g(x) = x^2 \cos 20\pi x$, and $h(x) = x^2$ on the same screen.

26. Use the Squeeze Theorem to show that

$$\lim_{x \rightarrow 0} \sqrt{x^3 + x^2} \sin \frac{\pi}{x} = 0$$

Illustrate by graphing the functions f , g , and h (in the notation of the Squeeze Theorem) on the same screen.27. If $4x - 9 \leq f(x) \leq x^2 - 4x + 7$ for $x \geq 0$, find $\lim_{x \rightarrow 4} f(x)$.